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The second approximation of the asymptotic phase for the Yukawa potential, treated with Laplace-transformations

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With 6 figures in the text

Summary

The Schrödinger equation is here solved with the aid of an expansion in the coupling-constant. Every term in the series is written as a Laplace-integral, and from this the asymptotical behaviour of the solution is easily obtained. The asymptotic phase will then also be expressed as a power series in the coupling-constant. The first term corresponds to the usual Born's approximation, the higher terms can be considered as corrections for this formula. In this paper the explicit calculations are carried out only to the second approximation and for small angular momenta (S- P- and D-states). The numerical results are compared with earlier approximate formulae of PAIS and RAMSEY and with the more accurate calculations made by the variational method of HULTHÉN.

General formulae

If the interaction potential between two particles is called $U(r)$, the radial part R of the wave function satisfies the equation

$$\frac{d^2}{dr^2}(rR) + \left[k^2 - \frac{l(l+1)}{r^2} + U(r)\right] Rr = 0. \tag{1}$$

For the following calculations it is more convenient to work with, instead of R, the function f defined by

$$R = f r^l \tag{2}$$

and satisfying the equation

$$r\left(\frac{d^2 f}{dr^2} + k^2 f\right) + 2(l+1)\frac{df}{dr} + U r f = 0. \tag{3}$$

As is well known, it is sufficient to consider only real solutions of this equation, as both the real and the imaginary parts of a complex solution have to satisfy the equation separately.

We now introduce the coupling constant λ, defined by

$$U(r) = \lambda \varphi(r) \tag{4}$$

where $r\varphi(r)$ is a dimensionless expression of the order of magnitude 1 (near the origin).

For the solution f we put

$$f = \sum_{n=0}^{\infty} \lambda^n f_n \tag{5}$$

and get from the equations (5) and (3) the following formula for the computation of f_n, when f_{n-1} is already known[1]

$$r\left(\frac{d^2 f_n}{dr^2} + k^2 f_n\right) + 2(l+1)\frac{d f_n}{dr} + r\varphi f_{n-1} = 0. \tag{6}$$

The general solution of (6) can be written

$$f_n = f_n^{(0)} + C_n f_0 \tag{7}$$

where $f_n^{(0)}$ is a particular solution of (6), C_n a constant of integration and f_0 the general solution of the homogeneous equation. The function f_0 is thus the first term in the series (5).

Given a system of functions $f_n^{(0)}$ satisfying (6) we compute the function

$$f^{(0)} = \sum_{n=0}^{\infty} \lambda^n f_n^{(0)} \tag{8}$$

which is obviously one solution of (6). If we now use the more general expressions (7), we get instead of $f_n^{(0)}$ the following system of functions

$$\begin{aligned} f_1 &= f_1^{(0)} + C_1 f_0 \\ f_2 &= f_2^{(0)} + C_1 f_1^{(0)} + C_2 f_0 \\ &\cdots\cdots\cdots\cdots \\ f_n &= f_n^{(0)} + C_1 f_{n-1}^{(0)} + \cdots + C_n f_0 \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned} \tag{9}$$

Equation (5) now gives the solution

$$f = \Sigma \lambda^n f_n^{(0)} + C_1 \Sigma \lambda^{n+1} f_n^{(0)} + \cdots = f^{(0)}(1 + C_1\lambda + C_2\lambda^2 + \cdots). \tag{10}$$

From equation (10) we see that we do not get any essentially new solution if we use constants C_n in (7), which are not zero. In what follows we are thus only interested in particular solutions of (6).

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[1] An expansion of this kind has earlier been used by HELLUND [11]. His series is, however, not convergent, compare FRÖBERG [12].

Let us suppose that we know the functions f_0, f_1, ..., f_{n-1} and that for the product $r\varphi f_{n-1}$ we have the integral representation

$$r\varphi f_{n-1} = \int_C e^{pr} h_{n-1}(p)\, dp \tag{11}$$

where C is some complex path of integration.

We put for the solution of (6)

$$f_n = \int_C h_{n-1}(p)\, V(p) \int e^{qr} W(q)\, dq\, dp \tag{12}$$

and get for the determination of $W(q)$ and $V(p)$ the following equation

$$V(p) \Big/ e^{qr}(q^2 + k^2)\, W(q) + \int e^{qr} \left[-\frac{dW}{dq}(q^2 + k^2) + 2lqW \right] dq = -e^{pr}. \tag{13}$$

This technique for solving the Schrödinger equation has previously been used by JOHANSSON [*1*][2], who has applied it to a radiation problem, and by SEXL [*2*], who has solved a special scattering problem in this way. Also in the mathematical literature *e.g.* WHITTAKER-WATSON [*3*], similar methods are frequently used.

We solve equation (13) by putting

$$-(q^2 + k^2)\frac{dW}{dq} + 2lqW = P(q) \tag{14}$$

where $P(q)$ is an arbitrary polynomial in q. Equation (14) can be integrated immediately, and we get

$$W(q) = (q^2 + k^2)^l \left\{ A \log\left(1 + \frac{q^2}{k^2}\right) + B \operatorname{arctg}\frac{q}{k} \right\} + R(q) \tag{15}$$

where A and B are constants and $R(q)$ is another polynomial in q. We now have

$$f_n(r) = \int_C h_{n-1}(p)\, V_n(p) \int \left\{ A_n \log\left(1 + \frac{q^2}{k^2}\right) + B_n \operatorname{arctg}\frac{q}{k} \right\} \cdot \\ \cdot e^{qr}(q^2 + k^2)^l\, dq\, dp \tag{16}$$

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[2] JOHANSSON has discussed the case of a hydrogen atom with its electron originally in the ground state and calculated the absorbtion coefficient for radiation due to scattering. The problem is first treated non-relativistically, though some relativistic calculations are also made. It is, however, not necessary to limit oneself to the ground state, and the calculations of JOHANSSON can without difficulty be carried out (non-relativistically) for an arbitrary state of the electron. In this way one gets, after a comparison with the ordinary perturbation theory, a comparatively simple deduction of the polar matrix-elements of the hydrogen atom — even for the continuous spectrum. The results agree with those of GORDON [Ann. d. Physik *2*, 1031 (1929)].

$n \neq 0$, and

$$f_0(r) = A_0 \int e^{pr} (p^2 + k^2)^l \log\left(1 + \frac{p^2}{k^2}\right) dp + \\ + B_0 \int e^{pr} (p^2 + k^2)^l \operatorname{arctg} \frac{p}{k}\, dp\,. \tag{16 a}$$

The integrands in (16) and (16 a) have singular points for $p = \pm\, i k$. As we wish to obtain a real solution, we have two possible paths of integration, shown in *Figs. 1* and *2*.

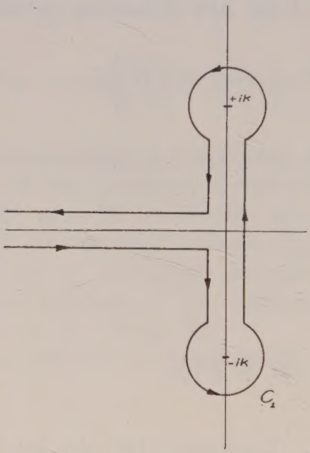

Fig. 1.

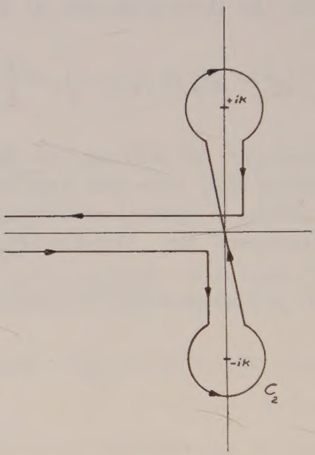

Fig. 2.

Obviously

$$\int_{C_1} f(p) \log\left(1 + \frac{p^2}{k^2}\right) dp = 2\,i \int_{C_2} f(p) \operatorname{arctg} \frac{p}{k}\, dp$$

and

$$\int_{C_2} f(p) \log\left(1 + \frac{p^2}{k^2}\right) dp = 2\,i \int_{C_1} f(p) \operatorname{arctg} \frac{p}{k}\, dp$$

($f(p)$ supposed regular in $p = \pm\, i k$). We may thus limit ourselves to considering only the path in *Fig. 2*. When this contour has been passed by the point p, $\log\left(1 + \frac{p^2}{k^2}\right)$ has resumed its original value and $\operatorname{arctg} \frac{p}{k}$ has increased by 2π. In the first term of equation (16 a) the path of integration can thus be modified as shown in *Fig. 3*. This solution is hence finite for all values of r. In the second term the contour cannot be simplified in this way and the integral diverges for $r = 0$. Accordingly we put $B_0 = 0$ and get

$$f_0(r) = A_0 \int e^{pr} (p^2 + k^2)^l \log\left(1 + \frac{p^2}{k^2}\right) dp. \tag{17}$$

For large values of r we get by a well-known method (compare *e.g.* [*3*] p. 343).

$$f_0(r) \to -4\pi A_0 \, l! \, (2k)^l r^{-l-1} \sin\left(kr - \frac{l\pi}{2}\right). \tag{17 a}$$

The numerical value of the factor A_0 is fixed by the condition that the amplitude is to tend to unity for large r, and we get

$$A_0 = -\frac{(2k)^{-l}}{4\pi l!}. \tag{17 b}$$

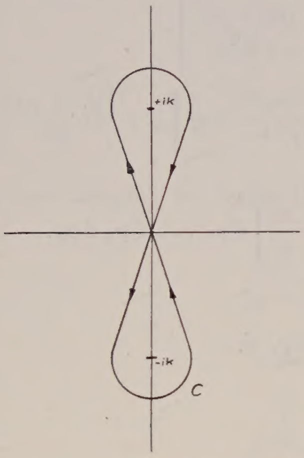

Fig. 3.

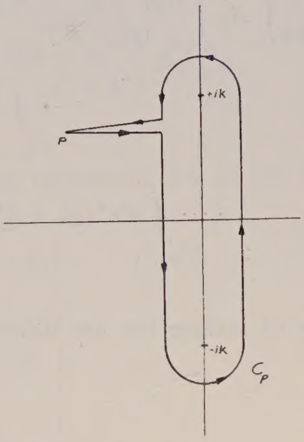

Fig. 4.

If n is not zero in equation (16), we choose the path C_p in *Fig. 4* and get from equation (13)

$$A_n V_n(p) = \frac{i}{4\pi} (p^2 + k^2)^{-l-1}. \tag{18}$$

The constant B_n in (16) corresponds to C_n in (7) and according to (10) we put $B_n = 0$ for all values of n.

Hence we get

$$f_n(r) = \frac{i}{4\pi} \int_C h_{n-1}(p) (p^2 + k^2)^{-l-1} \int_{C_p} e^{qr} (q^2 + k^2)^l \log\left[1 + \left(\frac{q}{k}\right)^2\right] dq \, dp. \tag{19}$$

Our next problem is to calculate the function $h_n(p)$ in equation (11), starting from equation (19). This is in the general case rather complicated but is an extremely simple problem if

$$\varphi(r) = \frac{e^{-\mu r}}{r}. \tag{20}$$

In this case we have

$$r\,\varphi(r)\,f_n(r) = \frac{i}{4\pi}\int\limits_{C} h_{n-1}(p)\,(p^2 + k^2)^{-l-1} \int\limits_{C_{p-\mu}} e^{qr}\,[(q+\mu)^2 + k^2]^l \cdot \log\left[1 + \frac{(q+\mu)^2}{k^2}\right] dq\,dp. \tag{21}$$

The equations (17), (19) and (21) give together

$$f_n(r) = \left(\frac{i}{4\pi}\right)^n A_0 \int\limits_{C} \frac{[(p_0+\mu)^2 + k^2]^l}{(p_0^2 + k^2)^{l+1}} \log\left[1 + \frac{(p_0+\mu)^2}{k^2}\right] \cdot \int\limits_{C_{p_0-\mu}} \frac{[(p_1+\mu)^2 + k^2]^l}{(p_1^2 + k^2)^{l+1}} \log\left[1 + \frac{(p_1+\mu)^2}{k^2}\right] \cdot \cdots \int\limits_{C_{p_{n-1}}} e^{p_n r}\,(p_n^2 + k^2)^l \log\left[1 + \frac{p_n^2}{k^2}\right] dp_n \ldots dp_1\,dp_0. \tag{22}$$

The paths of integration are illustrated in *Fig. 5*.

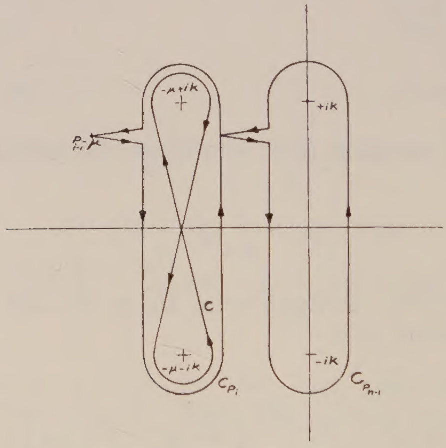

Fig. 5.

It is of course possible to choose the paths $C, \ldots, C_{p_{n-2}-\mu}$ in such a way that they surround more than two singular points but, as a rule, solutions obtained in this way will not be real.

The first (Born's) approximation

If $n = 1$, we get from formula (22)

$$f_1 = \frac{i}{4\pi} A_0 \int\limits_C \frac{[(p+\mu)^2 + k^2]^l}{(p^2+k^2)^{l+1}} \log\left[1 + \frac{(p+\mu)^2}{k^2}\right] \cdot \int\limits_{C_p} e^{qr} (q^2+k^2)^l \log\left(1 + \frac{q^2}{k^2}\right) dq\, dp. \tag{23}$$

For large values of r we get from this expression

$$f_1(r) \to \frac{1}{2i} \frac{\cos\left(kr - \frac{l\pi}{2}\right)}{r^{l+1}} \int\limits_{-\mu-ik}^{-\mu+ik} \frac{[(p+\mu)^2 + k^2]^l}{(p^2+k^2)^{l+1}} dp. \tag{24}$$

If in the last integral we make the following transformation of the integration variable

$$\left.\begin{aligned} \frac{p+ik}{p-ik} &= \alpha(x-\beta) \\ \alpha &= \frac{2k^2}{\mu^2} \frac{\mu}{\mu+2ik} \\ \beta &= -1 - \frac{\mu^2}{2k^2} \end{aligned}\right\} \tag{25}$$

we get

$$\int\limits_{-\mu-ik}^{-\mu+ik} \frac{[(p+\mu)^2 + k^2]^l}{(p^2+k^2)^{l+1}} dp = \frac{i}{k}\left(-\frac{1}{2}\right)^{l+1} \int\limits_{-1}^{+1} \frac{(1-x^2)^l}{(\beta-x)^{l+1}} dx = \frac{i}{k} Q_l(-\beta) \tag{26}$$

where $Q_l(-\beta)$ is a Legendre function of the second kind ([*3*] p. 316). Combining the equations (24) and (26) we get the asymptotic form of the function $f_1(r)$

$$f_1(r) \to \frac{1}{2k} Q_l\left(1 + \frac{\mu^2}{2k^2}\right) \frac{\cos\left(kr - \frac{l\pi}{2}\right)}{r^{l+1}}. \tag{27}$$

This formula has earlier been proved by HULTHÉN [*4*] in a slightly different way.

The second approximation

We here first discuss the S-states. In this case we get as the asymptotic form of (22)

$$f_2 \to \frac{i}{4} \frac{\cos kr}{kr} \int_{-\mu-ik}^{-\mu+ik} \frac{dp}{p^2+k^2} \cdot \left[2 \operatorname{arctg} \frac{p-\mu}{k} - \operatorname{arctg} \left(-i - \frac{\mu}{k} \right) - \operatorname{arctg} \left(i - \frac{\mu}{k} \right) \right]. \tag{28}$$

To make the arctg-functions and the logarithms unique, we make a cut along the real positive axes of the complex p-plane and, after another integration, obtain

$$f_2 \to \frac{\cos kr}{r} \frac{1}{4k^2} \left[\frac{\vartheta + \pi}{2} \log \left(1 + 4 \frac{k^2}{\mu^2} \right) + 2ik \int_{-\mu-ik}^{-\mu+ik} \operatorname{arctg} \frac{p-\mu}{k} \frac{dp}{p^2+k^2} \right]. \tag{29}$$

$$\vartheta = \operatorname{arctg} \frac{2k}{\mu}. \tag{30}$$

The substitution $\dfrac{p+ik}{p-ik} = x$ here gives

$$2ik \int_{-\mu-ik}^{-\mu+ik} \frac{dp}{p^2+k^2} \operatorname{arctg} \frac{p-\mu}{k} = \frac{i}{2} \left\{ \log \left(1 + 4 \frac{k^2}{\mu^2} \right) \log \left(\frac{\mu}{2ik-\mu} \right) + \int_{\frac{\mu}{\mu+2ik}}^{\frac{\mu-2ik}{\mu}} \frac{dx}{x} \left[\log \left(x - \frac{\mu+2ik}{\mu} \right) - \log \left(x - \frac{\mu}{\mu-2ik} \right) \right] \right\}. \tag{31}$$

Putting (compare [*3*] p. 280)

$$\phi(x) = \int_1^x \frac{\log(1+x)}{x} dx \tag{32}$$

we get

$$\int \frac{dx}{x} \log \left(x - \frac{\mu+2ik}{\mu} \right) = \log \left(-\frac{\mu+2ik}{\mu} \right) \log \left(1 + 4 \frac{k^2}{\mu^2} \right) + \phi \left(-\frac{\mu-2ik}{\mu+2ik} \right) - \phi \left(-\frac{\mu^2}{(\mu+2ik)^2} \right)$$

and

$$\int \frac{dx}{x} \log\left(x-\frac{\mu}{\mu-2ik}\right) = \log\left(-\frac{\mu}{\mu-2ik}\right)\log\left(1+4\frac{k^2}{\mu^2}\right) - \\ -\phi\left(-\frac{\mu-2ik}{\mu+2ik}\right)+\phi\left(-\frac{(\mu-2ik)^2}{\mu^2}\right).$$

From the definition (32) it is easily proved that

$$\phi(x)+\phi\left(\frac{1}{x}\right)=\frac{1}{2}[\log x]^2 \tag{33}$$

(compare also [*5*] p. 344) and thus

$$2\,\mathrm{Re}\left\{\phi\left(-\frac{\mu-2ik}{\mu+2ik}\right)\right\}=\frac{1}{2}\left[\log\left(-\frac{\mu-2ik}{\mu+2ik}\right)\right]^2. \tag{33 a}$$

Noting that

$$\mathrm{Im}\,\{\phi(-a\,e^{ib})\}=-\sum_{n=1}^{\infty}\frac{a^n}{n^2}\sin(nb) \tag{34}$$

we get after a few simple transformations of the equations (29)–(34)

$$f_2 \rightarrow \frac{\cos kr}{r}\,\frac{1}{4k^2}\left[\left(\frac{\pi}{2}-\vartheta\right)\log\left(1+4\frac{k^2}{\mu^2}\right)-\psi(2\vartheta)+\sum_{n=1}^{\infty}\frac{\sin(2n\vartheta)}{n^2\left[1+4\dfrac{k^2}{\mu^2}\right]^n}\right] \tag{35}$$

where

$$\psi(x)=\sum_{n=1}^{\infty}n^{-2}\sin(nx). \tag{35 a}$$

For comparatively large values of $\frac{k}{\mu}$ $\left(\frac{k}{\mu}>0.5\right)$ the series

$$\sum_{n=1}^{\infty}\frac{\sin(2n\vartheta)}{n^2\left(1+4\dfrac{k^2}{\mu^2}\right)^n} \tag{36}$$

converges sufficiently rapidly to be useful for numerical calculations. The function $\psi(x)$ has been tabulated by CLAUSEN [*6*] to an accuracy of 16 decimals.

If $\frac{k}{\mu}$ is small, it is more convenient to calculate the difference between (36) and (35 a) directly, using *e.g.* the formula

$$\sum\frac{\sin(2n\vartheta)}{n^2\left(1+4\dfrac{k^2}{\mu^2}\right)^n}-\psi(2\vartheta)= \\ =\log K\cdot\mathrm{arctg}\,\frac{K\sin(2\vartheta)}{1-K\cos(2\vartheta)}-\int_1^K\frac{\log y\,dy}{1+y^2-2y\cos(2\vartheta)} \tag{37}$$

where

$$K = \left(1 + 4\frac{k^2}{\mu^2}\right)^{-1}. \tag{37 a}$$

This formula can be proved if the complex integration in the equation (32) is done along the path shown in *Fig. 6.* The above difference is obviously the integral along the last straight part of the path. If we here use the absolute value of the integration variable as an independent quantity, we get

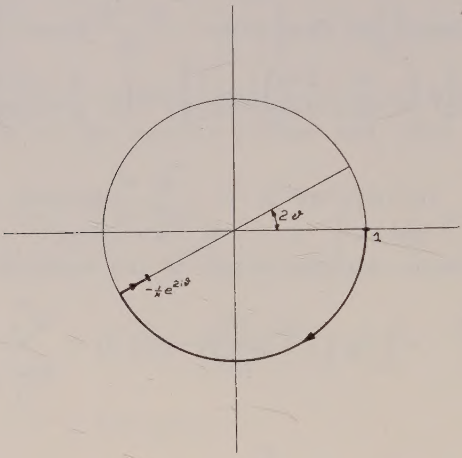

Fig. 6.

$$\int\limits_1^K \frac{dy}{y} \operatorname{arctg} \frac{y \cdot \sin(2\vartheta)}{1 - y\cos(2\vartheta)}.$$

An integration by parts enables us to transfer this integral to the right-hand side of equation (37). If $\frac{k}{\mu}$ is small ($K \approx 1$), the integral in (37) can easily be evaluated numerically. It must be observed, however, that for these small energies the first two terms in the series (5) give only a poor approximation of the wave function, unless the coupling constant is very small.

For the asymptotic phase of the S-states we now get the following expression

$$\operatorname{tg} \eta_S = \frac{\lambda}{4k}\log\left(1 + 4\frac{k^2}{\mu^2}\right) + \\ + \frac{\lambda^2}{4k^2}\left[\left(\frac{\pi}{2} - \vartheta\right)\log\left(1 + 4\frac{k^2}{\mu^2}\right) - \psi(2\vartheta) + \sum_{n=1}^{\infty}\frac{\sin(2n\vartheta)}{n^2\left(1 + 4\frac{k^2}{\mu^2}\right)^n}\right]. \tag{38}$$

This formula can be expected to give a good approximation of the asymptotic phase if the energy is large and the coupling constant is small. The following table gives the values of tg η computed with the formulae (27) and (38) compared with the values given by the variational formula of HULTHÉN [7].

$\frac{k}{\mu}$	$\frac{\lambda}{\mu} =$	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00
2.0	1st approx.	0.0885	0.171	0.256	0.35	0.43	0.51	0.60	0.68
	2nd approx.	0.0900	0.183	0.279	0.38	0.48	0.59	0.69	0.80
	numerical	0.0908	0.186	0.287	0.39	0.50	0.62	0.75	0.88
1.5	1st approx.	0.096	0.19	0.29	0.38	0.48	0.58	0.67	0.77
	2nd approx.	0.098	0.20	0.31	0.42	0.54	0.67	0.80	0.93
	numerical	0.098	0.22	0.35	0.50	0.65	0.80	0.94	1.08
1.0	1st approx.	0.101	0.201	0.302	0.40	0.50	0.60	0.70	0.80
	2nd approx.	0.105	0.219	0.342	0.47	0.62	0.77	0.92	1.09
	numerical	0.104	0.218	0.345	0.49	0.68	0.92	1.27	1.85
0.5	1st approx.	0.0866	0.173	0.26	0.35	0.43	0.52	0.61	0.69
	2nd approx.	0.0934	0.202	0.32	0.46	0.61	0.78	0.96	1.16
	numerical	0.0944	0.209	0.35	0.55	0.84	1.33	2.40	7.14

As a conclusion it must be said that the formula (38) gives a very poor approximation in those cases which are of practical interest.

For higher angular momenta we get instead of (28)

$$f_2 \to \frac{i}{4} \frac{\cos\left(k r - \frac{l\pi}{2}\right)}{r^{l+1}} \int\limits_{-\mu-ik}^{-\mu+ik} \frac{[(p+\mu)^2+k^2]^l}{(p^2+k^2)^{l+1}} \left\{ \int\limits_{-\mu-ik}^{p-\mu} + \int\limits_{-\mu+ik}^{p-\mu} \right\} \cdot \frac{[(q+\mu)^2+k^2]^l}{(q^2+k^2)^{l+1}}\, d q\, d p. \tag{39}$$

Using the transformation (25) the iterated integral can be written

$$-\frac{1}{k^2}\frac{1}{4^{l+1}} \int\limits_{-1}^{+1} \frac{(1-x^2)^l}{(\beta-x)^{l+1}} \left\{ \int\limits_{-1}^{\xi(x)} + \int\limits_{+1}^{\xi(x)} \right\} \frac{(1-y^2)^l}{(\beta-y)^{l+1}}\, d y\, d x \tag{40}$$

where

$$\left.\begin{aligned} \xi(x) &= \frac{\mu^2+4k^2}{2k^2}\,\frac{x-\beta-\lambda_1}{x-\beta-\lambda_2} \\ \lambda_1 &= \frac{\mu^2}{2k^2}\,\frac{\mu+2ik}{\mu-2ik} \\ \lambda_2 &= \frac{1}{2k^2}(\mu+2ik)^2 \end{aligned}\right\} \tag{41}$$

and

$$\beta = -1 - \frac{\mu^2}{2k^2}$$

as before.

In the formula (40) the y-integrations can always be carried out with the aid of elementary functions, but this is not so for the x-integration. Putting J_l for the whole integral in (40), it can, however, be seen that the difference

$$J_l - [P_l(\beta)]^2 J_0 \tag{42}$$

can always be expressed in logarithms, arctg- and rational functions. ($P_l(\beta)$ is here a Legendre function of the first kind). To prove this we have only to examine the coefficient of $\frac{\log(\xi-\beta)}{x-\beta}$ when the y-integrations have been made. This coefficient is

$$-\frac{1}{k^2}\,\frac{1}{4^{l+1}}\left[\frac{1}{l!}\,\frac{d^l}{d\beta^l}(1-\beta^2)^l\right]^2 = -\frac{1}{4k^2}[P_l(\beta)]^2$$

and hence the expression (42).

The remaining integrations are elementary but rather tedious. For the P-states we get

$$J_1 - \beta^2 J_0 = -ik^{-2}\left\{2\beta(\vartheta-\chi) + \frac{\mu^3}{2k^3}\log\left(1+\frac{k^2}{\mu^2}\right)\right\} \tag{43}$$

and for the D-states

$$J_2 - \frac{(3\beta^2-1)^2}{4}J_0 = -ik^{-2}\cdot \\ \cdot\left\{\frac{3\beta}{2}(3\beta^2-1)(\vartheta-\chi) + \frac{3}{8}\frac{\mu^3}{k^3}(3\beta^2-4\beta-3)\log\left(1+\frac{k^2}{\mu^2}\right) - \frac{3}{4}\frac{\mu^3}{k^3}\right\} \tag{44}$$

where

$$\chi = \operatorname{arctg}\frac{k}{\mu}. \tag{45}$$

The second approximation of the asymptotic phases will then be given by

$$\operatorname{tg}\eta_P = \frac{\lambda}{2k}\left[-\frac{\beta}{2}\log\left(1+4\frac{k^2}{\mu^2}\right)-1\right] + \\ + \frac{\lambda^2}{4k^2}\left[2\beta(\vartheta-\chi) + \frac{\mu^3}{2k^3}\log\left(1+\frac{k^2}{\mu^2}\right) + \beta^2\left\{\left(\frac{\pi}{2}-\vartheta\right)\log\left(1+4\frac{k^2}{\mu^2}\right) - \right.\right. \\ \left.\left. - \psi(2\vartheta) + \sum_{n=1}^{\infty}\frac{\sin(2n\vartheta)}{n^2\left(1+4\frac{k^2}{\mu^2}\right)^n}\right\}\right] \tag{46}$$

and

$$\operatorname{tg}\eta_D = \frac{\lambda}{2k}\left[\frac{3\beta^2-1}{4}\log\left(1+4\frac{k^2}{\mu^2}\right)+\frac{3}{2}\beta\right]+\frac{\lambda^2}{4k^2}\left[\frac{3}{2}\beta\,(3\beta^2-1)(\vartheta-\chi)+\right.$$

$$+\frac{3}{8}\frac{\mu^3}{k^3}(3\beta^2-4\beta-3)\log\left(1+\frac{k^2}{\mu^2}\right)-\frac{3}{4}\frac{\mu^3}{k^3}+ \tag{47}$$

$$\left.+\frac{(3\beta^2-1)^2}{4}\left\{\left(\frac{\pi}{2}-\vartheta\right)\log\left(1+4\frac{k^2}{\mu^2}\right)-\psi(2\vartheta)+\sum_{n=1}^{\infty}\frac{\sin(2n\vartheta)}{n^2\left(1+4\frac{k^2}{\mu^2}\right)^n}\right\}\right].$$

Formulae useful for the computation of the asymptotic phases in the higher states have been given by Pais [8] and Ramsey [9]. In his paper the latter author has given a number of applications, and by way of comparison the same cases have been evaluated numerically with the use of formulae (46) and (47). The result is shown in the following table.

l	$\frac{\lambda}{\mu}$	$\frac{k}{\mu}$	$\operatorname{tg}\eta_B$	$\operatorname{tg}\eta_P$	$\operatorname{tg}\eta_R$	$\operatorname{tg}\eta_{B^2}$	$\operatorname{tg}\eta_{\text{num}}$
1	−5.12	0.685	−0.341	−0.279	−0.253	−0.206	−0.256
1	−5.12	0.565	−0.253	−0.209	−0.187	−0.156	−0.190
1	−0.84	0.685	−0.0560	−0.0538	−0.0527	−0.0530	—
2	2.52	0.685	0.0348	0.0360	0.0366	0.0368	—

Here $\operatorname{tg}\eta_B$ and $\operatorname{tg}\eta_{B^2}$ are the first and second approximations used in this paper. $\operatorname{Tg}\eta_P$, $\operatorname{tg}\eta_R$ and $\operatorname{tg}\eta_{\text{num}}$ are the values given by the formulae of Pais, Ramsey and by numerical calculations. (These last three columns have been taken from the paper of Ramsey).

Numerical calculations for the P-states have also been made by Hansson [10] and we are appending a comparison with one of his results, too.

$\frac{k}{\mu}=0.8$

$\frac{\lambda}{\mu}$	−0.9	2.7	−4.5
1st approx.	−0.0736	0.221	−0.368
2nd approx.	−0.0685	0.267	−0.241
Hansson	−0.0691	0.291	−0.287

In these applications it has turned out that the coefficient of $\frac{\lambda^2}{4k^2}$ in (46) and (47) is given as a difference between comparatively large numbers. In the D-state, for example, this coefficient is 0.000580, but comes out as the difference between 14.752872 and 14.752292. This means that the sums (35 a) and (36) must be computed with a much greater accuracy than might at first be expected. The reason for this can be inferred if we compare the formulae (39) and (42). From (39) it is seen that the expansion of J_l in ascending powers of k begins with k^{2l+1}. The term $[P_l(\beta)]^2 J_0$, on the other hand, contains k^{-2l+1}, and hence the first powers of k must cancel in the end formula.

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